On the use of the proximity force approximation for deriving limits to short-range gravitational-like interactions from sphere-plane Casimir force experiments

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We discuss the role of the proximity force approximation in deriving limits to the existence of Yukawian forces—predicted in the submillimeter range by many unification models—from Casimir force experiments using the sphere-plane geometry. Two forms of this approximation are discussed, the first used in most analyses of the residuals from the Casimir force experiments performed so far, and the second recently discussed in this context in R. Decca *et al.* [Phys. Rev. D **79**, 124021 (2009)]. We show that the former form of the proximity force approximation overestimates the expected Yukawa force and that the relative deviation from the exact Yukawa force is of the same order of magnitude, in the realistic experimental settings, as the relative deviation expected between the exact Casimir force and the Casimir force evaluated in the proximity force approximation. This implies both a systematic shift making the actual limits to the Yukawa force weaker than claimed so far, and a degree of uncertainty in the α-λ plane related to the handling of the various approximations used in the theory for both the Casimir and the Yukawa forces. We further argue that the recently discussed form for the proximity force approximation is equivalent, for a geometry made of a generic object interacting with an infinite planar slab, to the usual exact integration of any additive two-body interaction, without any need to invoke approximation schemes. If the planar slab is of finite size, an additional source of systematic error arises due to the breaking of the planar translational invariance of the system, and we finally discuss to what extent this may affect limits obtained on power-law and Yukawa forces.

I. INTRODUCTION

Several unification schemes merging gravity and the standard model of strong and electroweak interactions predict the existence of short-range forces with coupling strength of the order of Newtonian gravity [1]. Efforts to evidence a *fifth force* have been envisaged regardless of any concrete unification scheme since various decades [2,3], and there are compelling reasons to improve our limits especially in the largely unexplored submillimeter range. Constraints in both coupling and range for these interactions have been obtained with various experimental setups, including the recent configurations using a disk-shaped torsional balance parallel to a rotating flat surface [4–7], or micromechanical resonators in a parallel-plane geometry [8–13]. Because of the surge of activity in the study of Casimir forces, limits have been also given in the submicrometer range based on the level of accuracy between Casimir theory and experiment. However, unlike the case of experiments performed between bodies kept at larger distances, the use of the parallel-plane geometry on such small length scale has been proven to be challenging in terms of parallelism [14–16], and therefore the attention has been focused on the analysis of the residuals in the Casimir theory-experiment comparison involving the sphere-plane configuration.

Dedicated efforts to obtain limits from sphere-plane Casimir experiments have involved the use of the so-called proximity force approximation (PFA) [17–23], which allows one to map the force F_{sp} between a sphere of radius R and a plane located at a distance a from the sphere into the energy per unit area E_{pp} of the parallel plate configuration, namely, $F_{\mathrm{sp}}(a) = 2\pi R E_{\mathrm{pp}}(a)$ [24]. This approximation is believed to be valid in the limit $a \ll R$ and to hold with a high degree of accuracy for forces between entities concentrated on the surfaces, such as electrostatic or Casimir forces between conductors [25–27]. Obviously, in order to test how well PFA approximates the exact force, one needs either to compute the interaction exactly or at least to assess reliable bounds. For the electrostatic sphere-plane interaction, the exact analytical result for the force is well known and has a closed form [28], such that deviations from PFA can be readily analyzed. For the Casimir sphere-plane interaction, the exact force has been computed only very recently, both for ideal [29–31] and real metallic plates [32]. Available analytical and numerical results seem to indicate that, at least for zero temperature and within the used plasma model, deviations from PFA applied to the sphere-plane Casimir interaction are small, of the order of 0.1% or higher, in recent Casimir experiments aiming to put limits to Yukawa interactions.

It has been argued in [33] that the application of the PFA to forces acting between entities embedded in volumetric manifolds, such as gravitational forces or their putative short-range components, is in general *invalid* and has to

be carefully scrutinized in each specific configuration. Based on this suggestion, a recent reanalysis of the PFA in the case of gravitational and Yukawian forces has been discussed in [34]. The main conclusion of this reanalysis is that "a confusion with different formulations of the PFA" existed in the previous literature, and that "care is required in the application of the PFA to gravitational forces." This confusion is stated to originate from a specific form of the PFA used so far, to be contrasted with a more general formulation of the PFA. In [34] it is also claimed that the difference between the two PFAs is negligible in the actual configuration used to give the allegedly best limits obtained in the 100 nm range [21,22].

In this paper we further discuss the meaning of the PFA in the case of volumetric forces. We argue that the discrepancy between the two forms of the PFA is a significant source of error in the determination of bounds on parameters of Yukawian forces from force residuals in Casimir sphere-plane experiments performed so far that used PFA to model such non-Newtonian forces. We then show that the general form for the PFA discussed in [34] is simply a different choice of the infinitesimal volume for integrating the force due to an extended object, and coincides with the exact result only in the case when one of the two surfaces is an infinite planar slab (or semispace). The level of approximation in using the two PFAs for Yukawa forces in the sphere-plane geometry is of the same order of magnitude as the Casimir theory-experiment comparison that uses PFA to compute the sphere-plane Casimir force (as already noticed in [34]). Therefore, since such a comparison provides force *residuals* that are in turn compared against the theory of Yukawa forces to obtain limits on its α-λ parameter space, the use of these *subsequent* PFA approximations of comparable level of approximation provides a possible source of systematic error, not carefully accounted for so far. We also argue that other volumetric effects not directly related to the PFA, such as the finite size of the planar surface used in the actual experiments, may provide a source of systematic error not taken into account so far, which strongly affects the limits to power-law forces [23,35], but should not be a major source of concern on limits to Yukawa forces. We believe that, considering the various number of complications related to the sphere-plane geometry, upgraded versions of parallel plate experiments such as the ones discussed in [8–13] could provide limits on Yukawian and power-law forces in the submicrometer range more immune to a set of systematic errors characteristic of the sphere-plane configuration.

II. PROXIMITY FORCE APPROXIMATIONS AND VOLUMETRIC FORCES BETWEEN EXTENDED OBJECTS

In order to introduce the notation and as a prelude to our discussion, we briefly summarize the results contained in [34]. The actual experimental configuration used in [21] is not a parallel plate geometry, rather it is a sphere-plane geometry, and the PFA is used to map the force between a sphere and a plane F_{sp} into the energy per unit area of the parallel plate configuration E_{pp}

$$F_{\mathrm{sp}}(a) = 2\pi\bar{R}E_{\mathrm{pp}}(a), \tag{1}$$

where $\bar{R} = \sqrt{R_x R_y}$ is the geometric average of the principal radii of curvature of the spherical surface evaluated at its point of minimum distance from the plane. In the experiment reported in [21], the force is measured by looking at the frequency shift of a mechanical resonator, as customary in atomic force microscopy [36], and as first reported in the context of Casimir force measurements in [37]. The frequency shift is proportional to the gradient of the force, and therefore

$$\Delta\nu^2 = \frac{1}{4\pi^2 m}\frac{\partial F_{\mathrm{sp}}}{\partial a} = \frac{\bar{R}}{2\pi m}\frac{\partial E_{\mathrm{pp}}}{\partial a} = \frac{\bar{R}}{2\pi m}P_{\mathrm{pp}}, \tag{2}$$

where P_{pp} is the plane-plane pressure, and m is the mass of the resonator. The measure of the frequency shift can then be mapped, via use of Eq. (1), into the equivalent pressure exerted between two fictitious parallel plates mimicking the actual sphere-plane geometry. Within the validity of Eq. (1), this is a valid assumption for the case of forces acting between surfaces, such as electrostatic forces between conductors or Casimir forces.

A first sign of the fact that there can be issues with the PFA in dealing with volumetric forces, such as the hypothetical Yukawian forces of gravitational origin, is manifested by noticing that the exact formula for the Yukawa force between two infinite parallel slabs depends on the thicknesses of both slabs, which implies that the PFA formula applied to the volumetric Yukawa force in the sphere-slab configuration also depends on *both* thicknesses (and on the sphere radius). However, the exact sphere-slab force obviously depends only on the slab thickness and on the sphere radius—it does not, and cannot, depend on the thickness of the *metaphysical* slab introduced in the virtual mapping to the parallel geometry. Indeed, consider the Yukawa potential energy for two pointlike masses m_1 and m_2, located at positions $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively,

$$U_{\mathrm{Yu}}(\mathbf{r}_1, \mathbf{r}_2) = -\alpha G m_1 m_2 \frac{e^{-|\mathbf{r}_2 - \mathbf{r}_1|/\lambda}}{|\mathbf{r}_2 - \mathbf{r}_1|}, \tag{3}$$

where, as usual, the strength of the Yukawa interaction is parametrized in terms of Newton's gravitational constant G through a dimensionless quantity α, and λ is its range. Assuming that the Yukawa interaction is additive, once integrated over two infinite, homogeneous parallel slabs separated by a distance a, one derives the corresponding pressure P_{Yu},

$$P_{\mathrm{Yu}}(a) = -2\pi\alpha G\rho_1\rho_2\lambda^2 e^{-a/\lambda}(1 - e^{-D_1/\lambda}) \\ \times (1 - e^{-D_2/\lambda}), \tag{4}$$

where D_1 and D_2 indicate the thickness of each slab, and ρ_1 and ρ_2 their densities. The exact Yukawa interaction in the sphere-slab geometry can be readily computed, assuming additivity. The result is [34]

$$F_{\text{Yu}}^{\text{exact}}(a) = -4\pi^2 \alpha G \rho_1 \rho_2 \lambda^3 R e^{-a/\lambda}(1 - e^{-D_1/\lambda})$$
$$\times (1 - \lambda/R + e^{-2R/\lambda} + e^{-2R/\lambda}\lambda/R). \quad (5)$$

As we mentioned above, most recent experimental works on limits to extra-gravitational forces from sphere-plane Casimir measurements used the usual PFA. In this approximation, the Yukawa force between a homogeneous sphere and an infinite homogeneous slab of thickness D_1 is

$$F_{\text{Yu}}^{\text{PFA}}(a) = 2\pi R P_{\text{Yu}}(a)$$
$$= -4\pi^2 \alpha G \rho_1 \rho_2 \lambda^3 R e^{-a/\lambda}(1 - e^{-D_1/\lambda})$$
$$\times (1 - e^{-D_2/\lambda}). \quad (6)$$

In this case one needed to consider, in order to map the actual sphere-plane configuration into a parallel plate geometry, a fictitious upper plate of thickness D_2 large enough, i.e. much larger than the explored Yukawa range $(D_2 \gg \lambda)$. Again, this situation may appear disturbing to whoever believes that any experiment-theory comparison should not rely on the introduction of arbitrary parameters not having a tangible, measurable counterpart in the concrete experimental setup. Clearly the PFA prediction Eq. (6) fails to give the exact result Eq. (5) in the range of its supposed validity, $a \ll R$, and it is necessary to assume $\lambda \ll R, D_2$ in order for PFA to tend to the exact result. But in this limit the volumetric nature of the interaction is lost, since the atoms in the ''bulk'' no longer contribute appreciably to the total force. Likewise, PFA fails to give the exact Newtonian interaction between the sphere and the slab, even in the range of its supposed validity $a \ll R$.

The authors of [34] consider the most general formulation of PFA [38], that we will call the ''exact'' PFA formula (EPFA) to distinguish it from the usual PFA approximation, described above. In the EPFA the force between two compact bodies is expressed as the sum of forces between plane parallel surface elements $dxdy$. The z component of the force is

$$F_z^{\text{EPFA}}(a) = \iint_\sigma dxdy P(x, y, z(x, y)), \quad (7)$$

where $P(x, y, z(x, y))$ is the pressure between two parallel plates at a local distance $z(x, y) = z_2(x, y) - z_1(x, y) > 0$ [$z_i(x, y)$ are the surfaces of the two bodies], a is the distance between them [smallest value of $z(x, y)$], and σ is the part of the (x, y) plane where both surfaces are defined (see Fig. 1). The EPFA prediction for the Yukawa force between a sphere and an infinite planar slab is [34]

$$F_{\text{Yu}}^{\text{EPFA}}(a) = -4\pi^2 \alpha G \rho_1 \rho_2 \lambda^3 R e^{-a/\lambda}(1 - e^{-D_1/\lambda})$$
$$\times (1 - \lambda/R + e^{-2R/\lambda} + e^{-2R/\lambda}\lambda/R), \quad (8)$$

which coincides with the exact result of Eq. (5). In Fig. 2 we plot the ratio η

$$\eta = \frac{F_{\text{Yu}}^{\text{EPFA}}}{F_{\text{Yu}}^{\text{PFA}}} = \frac{1 - \lambda/R + e^{-2R/\lambda} + e^{-2R/\lambda}\lambda/R}{1 - e^{-D_2/\lambda}} \quad (9)$$

as a function of the Yukawa parameter λ, for different values of the sphere radius keeping fixed $D_2 \to \infty$ (left plot), and for different values of D_2 keeping fixed the sphere radius at $R = 150$ μm (right plot). Note that η is independent of the sphere-slab separation a. When $D_2 \gg \lambda$, as surely realized in the left plot, PFA always *overestimates* the EPFA result, i.e. $\eta < 1$ (similarly to how the PFA overestimates the exact Casimir force in the sphere-plane geometry). Note also that when $D_2 \gg \lambda$ the atoms in the bulk of the two bodies do not contribute appreciably to the Yukawa force, thereby making it effectively of a surface character (i.e., nonvolumetric), as in the case of Casimir or electrostatic forces. Instead, for values of $D_2 \simeq 10\lambda$ (or smaller) the volumetric character of the Yukawa interaction is manifest, and η is no longer less than 1 (right plot). In this case the PFA applied to Yukawa forces is invalid and in the data analysis one should at least declare the value of D_2 imagined for which the limits are assessed. Overestimating the Yukawa force leads to stronger limits for the coupling constant α for a given λ with respect to the
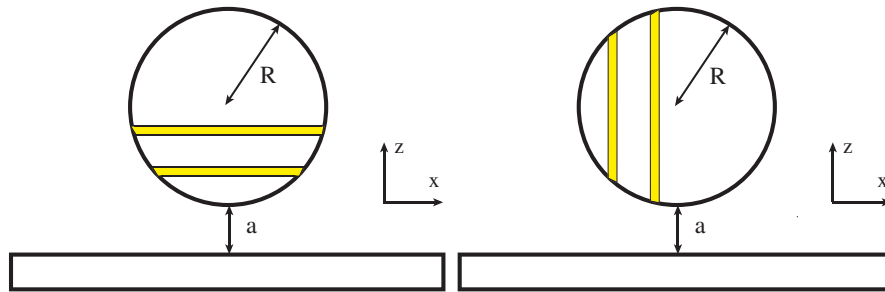

FIG. 1 (color online). Schematics of the integration for the sphere-slab configuration according to the usual slicing along horizontal infinitesimal slabs used for the exact calculation (left panel), and slicing using vertical columns (right panel) as used in the EPFA calculation. As far as the surface facing the sphere is an infinite plane (and therefore translational invariance of the potential V due to the plane being satisfied), these merely correspond to two different and equivalent choices for the integration volume.
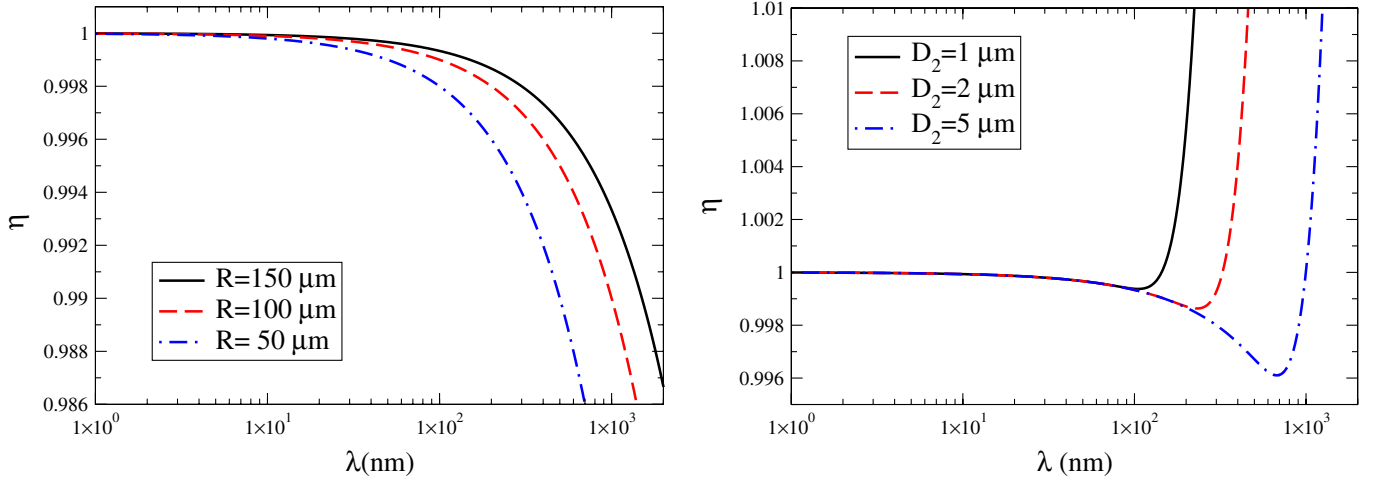
FIG. 2 (color online). Comparison between the EPFA and the PFA versus the range of the Yukawian force for a homogeneous sphere above an infinite homogeneous slab. (Left panel) Plot of η versus λ in the case of spheres of radius $R = 150$, 100, and 50 μm, in the limit of $D_2 \to \infty$. (Right panel) Same plot but for a sphere of radius 150 μm and finite values of the metaphysical parameter D_2 representing the thickness of the upper slab introduced in [34] for the comparison of the actual sphere-plane geometry to the parallel plate case using the PFA.

proper use of the EPFA. Considering the relatively small margins of improvement reported recently (see for instance Fig. 3 in [21]), a systematic shift due to the use of the PFA instead of the EPFA may lead to significant changes for the exclusion region in the α-λ plane. Both plots show that the use of PFA instead of EPFA is unreliable especially in the region near or above $\lambda = 100$ nm. Unfortunately the region in between 100 nm and a few μm is also the one *directly* explored with the Casimir force experiments, since actual measurements take place in this range of distance between the involved objects. It is known that the best limits on Yukawa interactions can be set for λ of the order of the actual explored distance between the two bodies $\simeq a$, and the extrapolation of the measurements to smaller λ is affected by the fast growth of the bounds as $\alpha \propto \exp(a/\lambda)$.

The fact that the EPFA Eq. (8) gives the correct exact results for the Yukawa (and also gravitational) force in the sphere-plane configuration is in fact a trivial consequence of the additivity of these interactions and of the translational invariance of the infinite plane surface, the shape of the second surface (in this case a sphere) being irrelevant. Indeed, the EPFA is just a *different* parametrization of the exact formula of the addition of forces between particles. On one hand, the exact interaction energy between a test mass and an infinite slab (or half-space) depends only on the normal coordinate z, being independent of the in-plane coordinates x, y by symmetry. Hence, the potential due to the infinite slab is $V(x, y, z) = V(z)$. For a body (e.g. a sphere) of mass density $\rho(x, y, z)$, additivity implies that the total interaction energy can be obtained as

$$U_{\text{body}} = \int dx dy dz \rho(x, y, z) V(z), \qquad (10)$$

and similarly for the force. Since V depends only on z, it is

convenient to compute the integral by adding forces at different slices at constant z, i.e. considering infinitesimal slices in z, then evaluating first the potential energy of a slice of the body parallel to the plane at a distance z

$$W(z) = \int dx dy \rho(x, y, z) V(z), \qquad (11)$$

and then integrating along z the quantity $W(z)$, one obtains the exact expression for the body-plane interaction U_{body} (see Fig. 1, left panel). On the other hand, the EPFA states that the interaction energy between the body and the plane is obtained from slicing the body into cylinders perpendicular to the plane (see Fig. 1, right panel) and integrating the cylinder-plane interaction energy along the portion of the body that faces the plane (i.e., one must integrate over the surface σ on the plane that is the normal shadow of the body). The potential energy of this column of the body centered around (x, y) is

$$G(x, y) = \int dz \rho(x, y, z) V(z), \qquad (12)$$

and then integrating along x, y the quantity $G(x, y)$, one gets the EPFA expression for the body-plane interaction U_{body} (see Fig. 1, right panel). Again, since the interaction is additive and it does not depend on the x, y coordinates, this integral is exactly equal to the previous one: we are simply integrating the same function $U(z)$ over the body using a different parametrization of the volumetric integral. The same holds for the force between any body (not necessarily a sphere) and an infinite slab (see also [39]).

However, when none of the two bodies is an infinite slab [e.g., two spheres of radii R_i, mass densities $\rho_i(x, y, z)$, separated by a distance a along the z direction], transla-

tional invariance along x-y is obviously broken, and the EPFA does not coincide with the exact formula. The exact interaction energy $U(x, y, z)$ between a source body and a test mass at position (x, y, z) can be easily computed. For instance, for the spherical source body U depends only on $r = (x^2 + y^2 + z^2)^{1/2}$, with the origin of coordinates at the center of the sphere. Integrating $U(x, y, z)$ over the volume of the second body one gets the exact result. For example, for the two-spheres case the gravitational energy depends only on the center-to-center distance and scales as $1/a$. Let us compare this known exact result with the EPFA prediction. One slices each body in cylindrical slabs, calculates the slab-slab interaction energy $U_{ss}(z)$ that depends on the local distance z between the slabs, and finally one adds up these contributions over the shadow of one of the bodies on the other one. It is clear that EPFA cannot give the exact result since U_{ss} is translational invariant but the exact U is not, and EPFA fails to predict the exact $1/a$ dependency. Therefore the EPFA formula for the energy and force of additive two-body interactions is a trivial reparametrization of the exact result when one of the bodies is an infinite plane (or slab). For other geometries, EPFA fails to give the correct result as a consequence of the broken translational invariance. In particular, this is the case of a sphere above a finite-size slab, like in experiments, especially those involving slabs of typical sizes comparable to those of the sphere.

In the experimental configuration used in [21] various substrates are present on the sphere and on the slab. Imagining that the layered slab is infinitely long, the Yukawa potential at a distance z from the top layer due to the slab is

$$V_{Yu}^\Delta(z) = -2\pi\alpha G\lambda^2 e^{-z/\lambda}[\rho_1'' e^{-\Delta_1''/\lambda}(e^{\Delta_1''/\lambda} - 1)$$
$$+ \rho_1' e^{-(\Delta_1''+\Delta_1')/\lambda}(e^{\Delta_1'/\lambda} - 1)$$
$$+ \rho_1 e^{-(\Delta_1''+\Delta_1'+D_1)/\lambda}(e^{D_1/\lambda} - 1)]. \quad (13)$$

Here Δ_1'' and ρ_1'' are the thickness and density of the top

layer, Δ_1' and ρ_1' are the thickness and density of the middle layer, and D_1 and ρ_1 and the thickness and density of the lower part of the layered slab. The last factor in Eq. (13) can be considered a sort of *effective* density of the planar surface, in which the various densities are weighted by their thicknesses in units of λ (indeed yielding their arithmetic average in the case of Δ', Δ'', and $D_1 \ll \lambda$).

We can compute the exact expression for the Yukawa interaction energy between the layered infinite slab and a layered sphere of mass density $\rho_2(x, y, z)$ using Eq. (10). As discussed above, this exact computation will trivially coincide with the EPFA expression. Let R and ρ_2 be the radius and density of the sphere, Δ_2' and ρ_2' the width and density of the inner layer on the sphere, Δ_2'' and ρ_2'' the width and density of the outer layer, and a the distance from the outer layer of the sphere to the top of the layered slab. The total EPFA Yukawa interaction energy can be written as a sum of contributions from each layer on the sphere, $U_{Yu}^{\Delta,EPFA} = U_2^\Delta + U_2'^\Delta + U_2''^\Delta$, where

$$U_2^\Delta = 2\pi\rho_2 \int_0^R r^2 dr \int_0^\pi d\theta \sin\theta \rho_2 V_{Yu}^\Delta(z),$$

$$U_2'^\Delta = 2\pi\rho_2' \int_R^{R+\Delta_2'} r^2 dr \int_0^\pi d\theta \sin\theta \rho_2' V_{Yu}^\Delta(z),$$

$$U_2''^\Delta = 2\pi\rho_2'' \int_{R+\Delta_2'}^{R+\Delta_2'+\Delta_2''} r^2 dr \int_0^\pi d\theta \sin\theta \rho_2'' V_{Yu}^\Delta(z).$$

Note that, instead of using horizontal or vertical slicings for the volume integration as done in Fig. 1 for the non-layered case, we use spherical slicings more appropriate for the layered sphere case. Here $z = a + \Delta_2'' + \Delta_2' + R - r\cos\theta$ denotes the vertical position of any infinitesimal mass element inside the layered sphere. Computing these integrals we obtain the EPFA expression for the Yukawa interaction energy between the layered infinite slab and the layered sphere

$$U_{Yu}^{\Delta,EPFA}(a) = -4\pi^2\alpha G\lambda^4 R e^{-(a+\Delta_2''+\Delta_2')/\lambda}\{\rho_1'' e^{-\Delta_1''/\lambda}(e^{\Delta_1''/\lambda} - 1) + \rho_1' e^{-(\Delta_1''+\Delta_1')/\lambda}(e^{\Delta_1'/\lambda} - 1)$$
$$+ \rho_1 e^{-(\Delta_1''+\Delta_1'+D_1)/\lambda}(e^{D_1/\lambda} - 1)\}\left\{\rho_2\left[1 - \frac{\lambda}{R} + e^{-2R/\lambda} + \frac{\lambda}{R}e^{-2R/\lambda}\right] + \rho_2'\left[\left(1 - \frac{\lambda}{R}\right)(e^{\Delta_2'/\lambda} - 1) + \frac{\Delta_2'}{R}e^{\Delta_2'/\lambda}\right.$$
$$+ e^{-2R/\lambda}\left(\left(1 - \frac{\lambda}{R}\right)(1 - e^{-\Delta_2'/\lambda}) + \frac{\Delta_2'}{R}e^{-\Delta_2'/\lambda}\right)\right] + \rho_2''\left[\left(1 - \frac{\lambda - \Delta_2'}{R}\right)e^{\Delta_2'/\lambda}(e^{\Delta_2''/\lambda} - 1) + \frac{\Delta_2''}{R}e^{(\Delta_2'+\Delta_2'')/\lambda}\right.$$
$$+ e^{-2R/\lambda}\left(\left(1 + \frac{\lambda + \Delta_2'}{R}\right)e^{-\Delta_2'/\lambda}(e^{-\Delta_2''/\lambda} - 1) + \frac{\Delta_2''}{R}e^{-(\Delta_2'+\Delta_2'')/\lambda}\right)\right]\right\}. \quad (14)$$

The EPFA expression for the corresponding force is $F_{Yu}^{\Delta,EPFA} = -\partial U_{Yu}^{\Delta,EPFA}/\partial a = \lambda^{-1} U_{Yu}^{\Delta,EPFA}$. Note that when there are no layers on the slab ($\Delta_1' = \Delta_1'' = 0$) and no layers on the sphere ($\Delta_2' = \Delta_2'' = 0$), then the expression for the force that follows from Eq. (14) is identical to Eq. (5). On the other hand, the PFA expression for the force

between the layered infinite slab and the layered sphere is $F_{Yu}^{\Delta,PFA} = 2\pi R P_{Yu}^\Delta$, where P_{Yu}^Δ is the pressure between two parallel layered slabs, one identical to the previous slab, and a metaphysical slab of width D_2 and density ρ_2, covered by two layers of widths and densities identical to the ones of the layered sphere above. Using Eq. (4) for the

various pairs of layers in the different slabs, we calculate the PFA expression for the layered sphere-slab force

$$
\begin{aligned}
F_{\mathrm{Yu}}^{\Delta,\mathrm{PFA}}(a) = {}& -4\pi^2 \alpha G \lambda^3 R e^{-a/\lambda} \{ \rho_1 (1 - e^{-D_1/\lambda}) [\rho_2 e^{-(\Delta_1' + \Delta_1'' + \Delta_2' + \Delta_2'')/\lambda}(1 - e^{-D_2/\lambda}) + \rho_2' e^{-(\Delta_1' + \Delta_1'' + \Delta_2')/\lambda}(1 - e^{-\Delta_2'/\lambda}) \\
& + \rho_2'' e^{-(\Delta_1' + \Delta_1'')/\lambda}(1 - e^{-\Delta_2''/\lambda})] + \rho_1'(1 - e^{-\Delta_1'/\lambda})[\rho_2 e^{-(\Delta_1'' + \Delta_2' + \Delta_2'')/\lambda}(1 - e^{-D_2/\lambda}) \\
& + \rho_2' e^{-(\Delta_1'' + \Delta_2'')/\lambda}(1 - e^{-\Delta_2'/\lambda}) + \rho_2'' e^{-\Delta_1''/\lambda}(1 - e^{-\Delta_2''/\lambda})] + \rho_1''(1 - e^{-\Delta_1''/\lambda})[\rho_2 e^{-(\Delta_2' + \Delta_2'')/\lambda}(1 - e^{-D_2/\lambda}) \\
& + \rho_2' e^{-\Delta_2''/\lambda}(1 - e^{-\Delta_2'/\lambda}) + \rho_2''(1 - e^{-\Delta_2''/\lambda})] \}. \tag{15}
\end{aligned}
$$

To assess the effect of the multilayered structures, we have evaluated the ratio η_Δ/η with $\eta_\Delta = F_{\mathrm{Yu}}^{\Delta,\mathrm{EPFA}}/F_{\mathrm{Yu}}^{\Delta,\mathrm{PFA}}$ and $\eta = F_{\mathrm{Yu}}^{\mathrm{EPFA}}/F_{\mathrm{Yu}}^{\mathrm{PFA}}$, as a function of λ for three radii of curvatures of the sphere, assuming for the PFA calculation a value of the metaphysical parameter $D_2 = 10^8 \ \mu m$ (see Fig. 3, left panel). The effect of multilayers is to slightly flatten η_Δ as compared to η in the homogeneous case (Fig. 2, left panel). The dependence of the same ratio for a fixed value of R and different values of the metaphysical parameter D_2 is shown in Fig. 3, right panel. Note that η_Δ is independent of the sphere-slab separation, just as η is.

As discussed in the Introduction, the PFA used in all recent sphere-plane Casimir experiments for the Casimir theory-experiment comparison is expected to approximate the exact Casimir force within 0.1%. This expectation comes from recent analytical approaches to the sphere-plane Casimir interaction [29–32] that, although formally exact, require the evaluation of the determinant of an infinite-dimensional matrix, which becomes a numerically demanding task, especially in the PFA regime, $a \ll R$, where larger and larger matrices are needed for convergence. Numerical computations of the exact, zero-temperature sphere-plane Casimir force using parameters

for metallic spheres ($R = 10 \ \mu m$ and optical response modeled by the simple plasma model with plasma wavelength $\lambda_p = 136$ nm) show that deviations from PFA can be as large as 20% for the smallest $a/R \approx 0.5$ studied numerically (see Fig. 2 of [32]). An extrapolation to smaller values of a/R using a cubic polynomial fit of the numerical data is also provided in [32]. Assuming one can use it for the recent Casimir sphere-plane experiment [21] (with a radius of curvature $R = 151.3 \ \mu m$) gives a deviation from PFA of the order of 0.1% at the smallest value of $a/R \approx 0.001$ reached in the experiment ($a_{\min} \approx 160$ nm). Since the limits to non-Newtonian forces are obtained using the residuals in the Casimir theory-experiment comparison, in order to meaningfully replace the exact formula of the Yukawa force with its PFA approximation, the level of accuracy between these two should be therefore a small fraction, for instance 10% of the accuracy with which the Casimir force is controlled by using PFA rather than the exact expression for the sphere-plane Casimir force. If this condition is not fulfilled, the derived limits could be off also by a large order of 100% correction. However, targeting a 10% accuracy level with respect to the Casimir theory-experiment accuracy implies deviations from $\eta = 1$
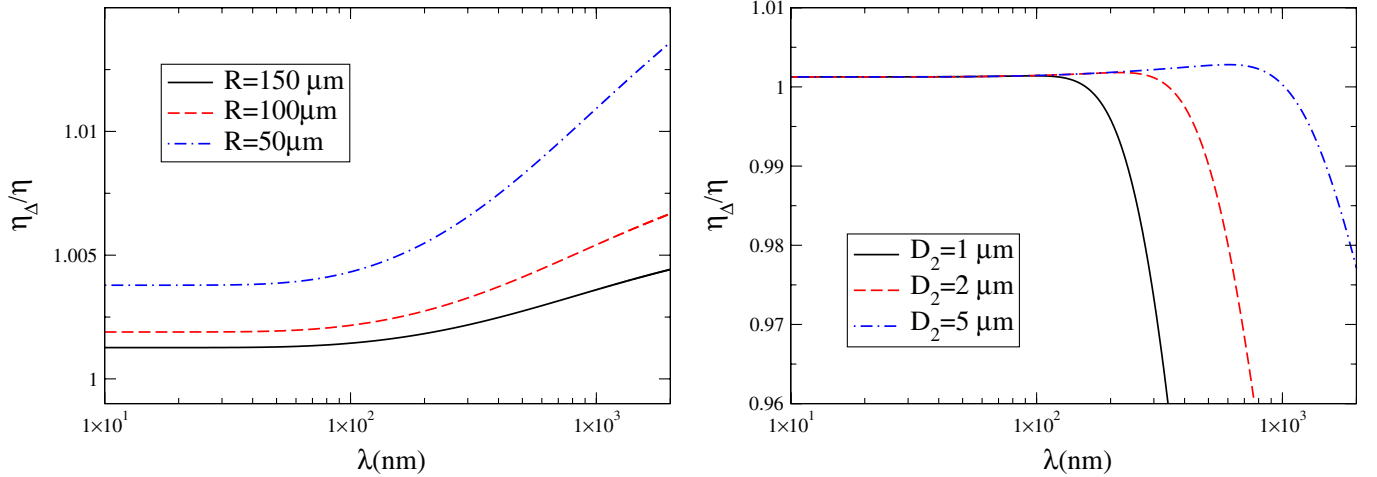

FIG. 3 (color online).   Ratio η_Δ/η for the comparison of the EPFA and PFA schemes for the multilayered and corresponding homogeneous situation [obtainable by using Eqs. (14) and (15) with $\Delta_2' = \Delta_2'' = 0$, $\rho_2' = \rho_2'' = 0$, and replacing R with $R + \Delta_2' + \Delta_2''$]. (Left panel) Ratio η_Δ/η versus the range of the Yukawian force for different values of the radius of the inner sphere. The parameters for the layered sphere are $\Delta_2' = 10$ nm, $\Delta_2'' = 180$ nm, $\rho_2 = 4.1$ g/cm^3, $\rho_2' = 7.14$ g/cm^3, and $\rho_2'' = 19.28$ g/cm^3. The parameters for the layered slab are $D_1 = 3.5 \ \mu m$, $\Delta_1' = 10$ nm, $\Delta_1'' = 210$ nm, $\rho_1 = 2.33$ g/cm^3, $\rho_1' = 7.14$ g/cm^3, and $\rho_1'' = 19.28$ g/cm^3. In the evaluation of the PFA force $F_{\mathrm{Yu}}^{\Delta,\mathrm{PFA}}$, a value of the metaphysical parameter $D_2 = 10^8 \ \mu m$ is used. (Right panel) Ratio η_Δ/η versus the range of the Yukawian force for different values of the metaphysical parameter D_2 and a radius of curvature of $R = 150 \ \mu m$.

of 0.01%, which can be obtained, as seen in Fig. 2, only in the range of λ below 100 nm. The presence of substrates with different densities tends to mitigate the discrepancy between the EPFA and the PFA, as seen by the curves in Fig. 3, but there is an irreducible systematic factor even at small λ. Indeed, in the limit $\lambda \rightarrow 0$, we have $\eta_\Delta \approx 1 + (\Delta_2' + \Delta_2'')/R$, that, in the case of the experiment reported in [21], is equal to 1.001 26, i.e. a correction already equal to 0.126%.

All these systematic sources of uncertainty could be even larger in experiments for which the radius of curvature of the sphere is not adequately optimized. Indeed, the use of spheres with smaller radius of curvature is affected more by this effect, as emphasized in the left plot of Fig. 2 and in Fig. 3 for the cases of $R = 50$ μm and 100 μm. Moreover, for small spheres the PFA approximation to the Casimir force itself is less accurate. The use of spheres with large radius of curvature is beneficial to reduce these sources of error in the experiment-theory comparison, but may face experimental issues recently identified in [40] and interpreted as due to deviations from an ideal spherical geometry (as proposed in [41]) and/or a consequence of larger sensitivity to electrostatic patch effects [33,40].

III. BREAKING THE X-Y TRANSLATIONAL INVARIANCE

In the previous section we have seen that translational invariance is crucial to make the EPFA reproduce the exact result, but in actual experiments such an invariance is obviously satisfied only approximately, leading to an additional source of systematic error related to the finite size of the surfaces, as we discuss here for both power-law forces, and for Yukawa forces. Instead of computing the more involved problem of the gravitational force between a sphere and a finite-size slab, we consider here the simpler case of the gravitational force acting on a pointlike test mass m_2 above the center of a disk of thickness D_1, radius R_d, and mass density ρ_1. We obtain

$$F_g(z_2) = -2\pi G\rho_1 m_2 \int_{-D_1}^{0} dz_1 \int_{0}^{R_d} r dr \frac{z_2 - z_1}{[r^2 + (z_2 - z_1)^2]^{3/2}}$$
$$= -2\pi G\rho_1 m_2 \{D_1 + (R_d^2 + z_2^2)^{1/2}$$
$$- [R_d^2 + (z_2 + D_1)^2]^{1/2}\}, \qquad (16)$$

where z_2 is the distance between the test mass and the disk. The force becomes independent of R_d only in the limit $R_d \gg D_1, z_2$ (in which case it is also independent of z_2). In order to assess the different forces acting on the various parts of a sphere in the presence of a disk of finite radius we evaluate the ratio between the forces exerted at the point of the sphere closest to the plane ($z_2 = a$) and the farthest point ($z_2 = a + 2R$). This quantity is simple to evaluate yet provides a practical figure of merit for how much the extended geometry of the sphere is affected by the finite size of the disk. This gives a ratio $\xi_g = F_g(z_2 = $

$a)/F_g(z_2 = a + 2R)$:

$$\xi_g = \frac{\beta + [\gamma^2 + \kappa^2]^{1/2} - [(\gamma + \beta)^2 + \kappa^2]^{1/2}}{\beta + [(2 + \gamma)^2 + \kappa^2]^{1/2} - [(2 + \gamma + \beta)^2 + \kappa^2]^{1/2}}, \qquad (17)$$

where we have defined $\beta \equiv D_1/R$, $\gamma \equiv a/R$, and $\kappa \equiv R_d/R$.

This is a large correction, of the order of 300%, if a disk of radius equal to twice the radius of the sphere ($R_d = 2R$), in a geometrical setting not dissimilar from the one used in [21], is considered. Since the experiment in [21] is anyway insensitive to the gravitational force, like any experiment performed in the micrometer range, this is not a major practical concern. However, in the case of a more generic power law such as $F_N = -K\rho_1 m_2/r^N$ we get

$$F_N(z) = \frac{2\pi K\rho_1 m_2}{(N - 1)(N - 3)}\{(z + D_1)^{3-N} - z^{3-N}$$
$$+ (R_d^2 + z^2)^{(3-N)/2} - [R_d^2 + (z + D_1)^2]^{(3-N)/2}\}, \qquad (18)$$

apart from the cases of $N = 1$ and $N = 3$ in which logarithmic integrations occur. In these two cases one obtains

$$F_{N=1}(z) = \frac{\pi K\rho_1 m_2}{2}\{(z^2 + R^2)\ln(z^2 + R_d^2)$$
$$- [(z + D_1)^2 + R_d^2]\ln[(z + D_1)^2 + R_d^2]$$
$$+ (z + D_1)^2 \ln(z + D_1)^2 - z^2 \ln z^2\}, \qquad (19)$$

which in the limit $R_d \rightarrow \infty$ becomes independent of z, and

$$F_{N=3}(z) = -\frac{\pi K\rho_1 m_2}{2}\ln\left[\frac{(z^2 + R_d^2)(z + D_1)^2}{z^2[(z + D_1)^2 + R_d^2]}\right], \qquad (20)$$

which in the limit $R_d \rightarrow \infty$ behaves as $\ln(1 + D_1^2/z^2)$. Notice that the fact that the force is independent on the distance from an infinite plane is only characteristic of forces scaling with the inverse square of the distance, such as the gravitational force, making the integration of the force trivially geometrical. In the general case $n \neq 2$, even in the situation of a sphere in front of an infinite plane, different points of the sphere will feel different forces, with the farthest point feeling smaller (larger) force for a power-law exponent larger (smaller) than 2, as a consequence of the interplay between the solid angle and the distance scaling of the force, which makes peculiar the $N = 2$ case as expressed by the Gauss law. This is shown in Fig. 4 (left panel) for the cases of $N = 1, 2, 3$, and 4, with the ratio ξ_N between the forces evaluated at the top and at the bottom of the sphere, and in Fig. 4 (right panel) by showing the same ratio versus the power-law exponent for different values of the radius of the disk. Therefore, when considering power-law forces as the ones discussed for instance around Eq. 2 in [23], one should then take carefully into account the finite size of the plane in deriving limits to these forces [42].
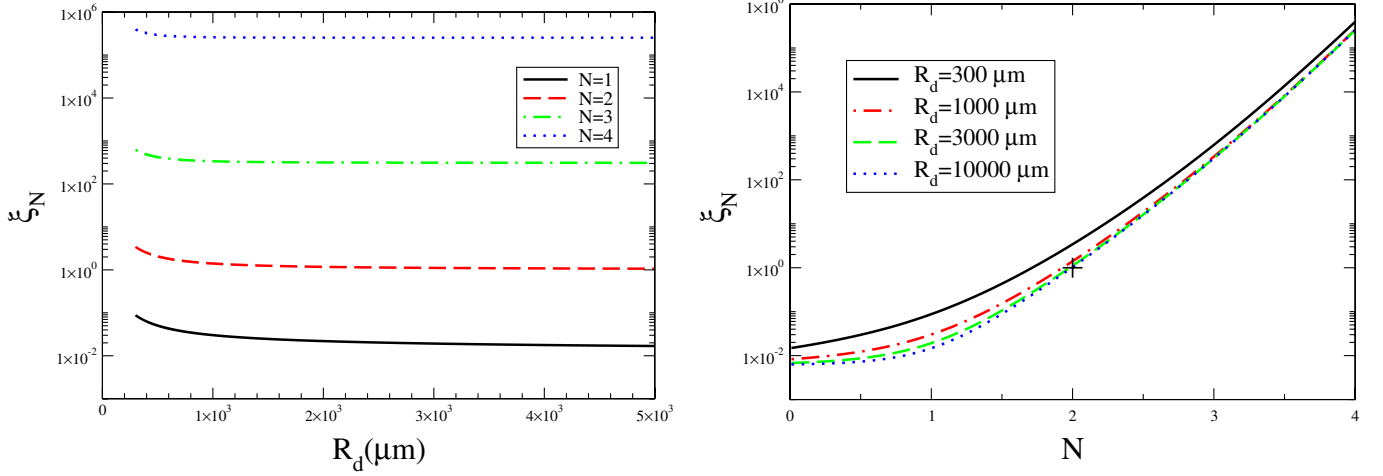
FIG. 4 (color online). (Left panel) Plot of the ratio ξ_N between the force $F(r) \propto r^{-N}$ exerted at the closest point of the sphere from the disk and the force exerted at the farthest point, versus the radius of the disk R_d constituting the planar surface of finite size, for four different exponents N. We assume a radius of the sphere equal to $R = 150 \ \mu m$, a sphere-plane distance of $a = 100$ nm, and that the center of the sphere is right above the center of the disk. Only the case of $N = 2$ (Newtonian gravitation) gives a ratio of unity in the large R_d limit. (Right panel) Plot of the ratio ξ_N versus the exponent of the power force law N for various radii of the disk R_d. The cross indicates the ratio $\xi = 1$ for the case of an exponent $N = 2$ and an infinite plane; it is provided as a help to the eye to better show the convergence in the case of $N = 2$ of ξ_N to unity with disks of progressively larger radii.

Finally, we discuss the effect of the finite size of the planar surface in the case of Yukawa forces. The potential energy of a pointlike particle of mass m_2 located at height z along the axis of a planar disk surface of radius R_d, density ρ_1, and thickness D_1, is

$$U_{Yu}(z) = -\alpha G \rho_1 m_2 \int_0^R dr r \int_0^{2\pi} \int_{-D_1}^0 [(z_2 - z_1)^2 + r^2]^{1/2}$$
$$\times e^{-\sqrt{(z_2-z_1)^2+r^2}/\lambda}$$
$$= -2\pi \alpha G \rho_1 m_2 \lambda^2 [e^{-z/\lambda}(1 - e^{-D_1/\lambda})$$
$$- e^{-\sqrt{z^2+R_d^2}/\lambda} + e^{-\sqrt{(z+D_1)^2+R_d^2}/\lambda}], \quad (21)$$

and the related force is

$$F_{Yu}(z) = -\frac{\partial U_{Yu}}{\partial z}$$
$$= -2\pi \alpha G \rho_1 m_2 \lambda e^{-z/\lambda}(1 - e^{-D_1/\lambda})$$
$$\times \left\{ 1 - \frac{z}{R_d} \frac{e^{-[1+(z/R_d)^2]^{1/2}R_d/\lambda+z/\lambda}}{[1 + (z/R_d)^2]^{1/2}(1 - e^{-D_1/\lambda})} \right.$$
$$\left. + \frac{z + D_1}{R_d} \frac{e^{-[1+((z+D_1)/R_d)^2]^{1/2}R_d/\lambda+z/\lambda}}{[1 + ((z+D_1)/R_d)^2]^{1/2}(1 - e^{-D_1/\lambda})} \right\},$$
$$(22)$$

where the finite-size terms appear as corrections to the indefinite plane formula originating by the first term alone. As before, we introduce as figure of merit the ratio $\xi_{Yu} = F_{Yu}(z_2 = a)/F_{Yu}(z_2 = a + 2R)$. This ratio is very large for realistic configurations, expressing the short-range nature of the force. Indeed, even in the infinite plane limit we have a ratio of $\xi_{Yu} = e^{2R/\lambda} \simeq e^{3000}$ in the case of a sphere of radius $R = 150 \ \mu m$ at $\lambda = 0.1 \ \mu m$. The dependence

on the disk radius become significant only at values of λ comparable to the radius of the sphere, as shown in Fig. 5. The presence of suppression factors for the farthest point of
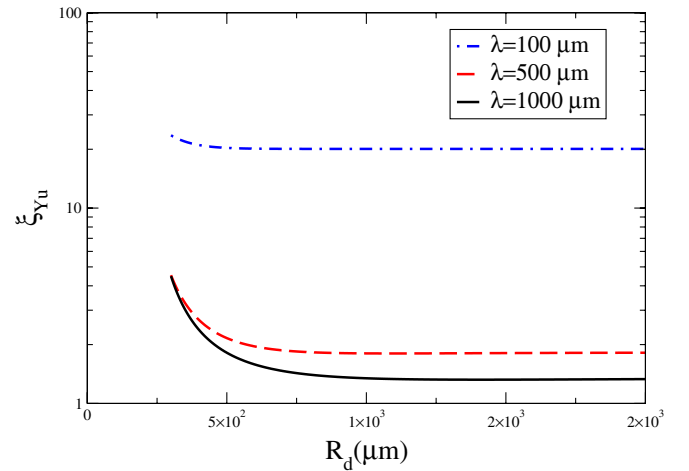

FIG. 5 (color online). Plot of the ratio ξ_{Yu} between the Yukawa force exerted at the closest point of the sphere from the disk and the force exerted at the farthest point, versus the radius of the disk R_d constituting the planar surface of finite size, for three different values of the Yukawa range λ. As before, we assume a radius of the sphere equal to $R = 150 \ \mu m$, a sphere-plane distance of $a = 100$ nm, and the center of the sphere right above the center of the disk. The values of λ are chosen to be 100, 500, and 1000 μm. In the last case the force may be considered as a long range one and the farthest point on the sphere is also contributing almost as much as the closest one. For λ progressively smaller than the radius of the sphere the ratio ξ_{Yu} gets larger and larger and the dependence on R_d is not appreciable.

the form $e^{2R/\lambda}$ makes the Yukawa force very insensitive to the finite size of the disk (for previous consideration, see also [43]).

IV. CONCLUSIONS

Our analysis, although confirming some of the results already discussed in [34], draws quite different conclusions from the common outcome. In particular, we argue that the application of PFA to volumetric forces is not rigorous if considered in its original formulation applied so far to evaluate the sphere-plane Yukawa interactions in the most accurate Casimir force experiments reported in [18–23]. Its application to volumetric forces is instead of a trivial nature if considered in the exact formulation EFPA discussed in [34], since the latter is identical to the exact calculation, just differing in the choice of the infinitesimal integration volume. We have shown that the usual PFA is an invalid approximation to compute volumetric forces. In particular, it does not reproduce in its usual range of validity exact known expressions for gravitational and Yukawa interactions in nontranslational invariant geometries, such as sphere-finite size slab or sphere-sphere configurations. The exact PFA is the exact expression for any additive two-body interaction when one of the bodies is translational invariant, as is the case for a sphere in front of an infinite homogeneous slab or half-space. For nontranslational invariant geometries, EPFA also fails to give the exact result for volumetric interactions, even in the regime of parameters where it is assumed to be valid, therefore also being an invalid approximation for volumetric forces.

The difference between the two formulations of the PFA is shown to affect significantly the limits obtained so far unless one considers a regime of Yukawa range so small, $\lambda \ll R, D_2$, that the approximation of a surface force (i.e. neglecting the Yukawa force due to the atoms in the bulk of the two bodies, therefore manifestly of nonvolumetric character) holds. By using the PFA the Yukawa force is overestimated and therefore the limits in the α-λ plane become more stringent than by using the exact force estimated via EPFA. Moreover, the use of PFA instead of the EPFA for the parameters of the experiment supposed to provide the strongest limits to Yukawian interactions [21] occurs with an accuracy of the same order of magnitude with which the exact Casimir force is expected to be also approximated by the corresponding PFA. On one hand, since the Casimir theory-experiment comparison provides force residuals that are in turn compared against the theory of Yukawa forces to obtain limits on them, the use of these subsequent PFA approximations of comparable level of

approximation provides a possible source of systematic error, not carefully accounted for so far. On the other hand, both Casimir and Yukawa PFAs overestimate the respective exact forces, and therefore the systematic source of error introduced by their use might be less critical than expected on first principles. In any case, it is important to assess as much as possible both sources of errors or, alternatively, to use exact expressions for the Yukawa and Casimir forces. Furthermore, a systematic source of error present even in the EPFA scheme for finite-size planar surfaces has been discussed and shown to be significant only for power-law forces. Our analysis suggests that future limits (or reanalysis of experiments already performed) on Yukawian forces should rely upon the use of the exact expression for the Yukawa force, as performed in [43,44].

It is also worth pointing out that all the above considerations hold provided that the simple scenario of additive forces is assumed, which is valid in general for weak forces among atoms in the low density limit, such that correlations leading to fluctuating forces are negligible. However, the hypothetical Yukawa forces should be located in a regime of coupling constants intermediate between the gravitational (additive) force and the Casimir (nonadditive) force. It is not understood *a priori* if the Yukawa force is weak enough to make the additivity assumption reliable, and this should be kept in mind in future broad-range searches of these forces.

Finally, considering the complications emerging in the sphere-plane geometry due to the presence of previously unidentified systematics such as the sensitivity to deviations from the ideal spherical geometry [33,40,41,45] and possible effects of variability of the contact potential with distance [40,45–49], it may be worth to focus future Casimir experiments to set bounds on extra-gravitational forces on the actual parallel-plane geometry, without the drawbacks of a virtual mapping from the sphere-plane geometry made explicit in this paper. The stronger force signal expected for the same distance between the two surfaces, the reduced sensitivity to distance-dependent contact potentials due to image charges, the absence of deviations from a uniform radius of curvature, the existence of exact mode summation techniques to compute the Casimir force, the possibility to control parallelism using recently developed technology [50], and the possibility to compensate off-line the lack of parallelism by using the PFA as discussed in [51], all point in the direction to continue this class of experiments in the actual parallel-plane configuration, extending below the 10 μm range the results of the experiments described in [4–13].

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